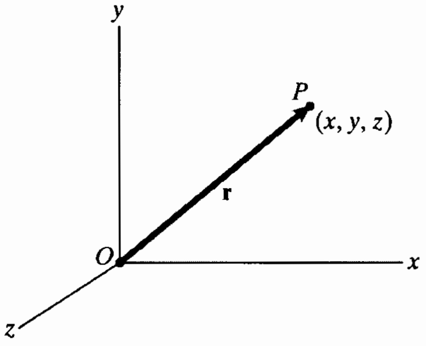
# Motion of a Point

$\mathbf{r}$ : position vector of $P$ relative to origin $O$. Then:

$\mathbf{v} = \dfrac{d\mathbf{r}}{dt}$ : velocity of $P$ relative to $O$, and

$\mathbf{a} = \dfrac{d\mathbf{v}}{dt}$ : acceleration of $P$ relative to $O$.



(a)

## Straight line motion

$$v = \frac{ds}{dt}, \qquad a = \frac{dv}{dt}$$

$$a = \frac{dv}{ds}v$$

If the acceleration is a function of v, use separation of variables:

$$\frac{dv}{dt} = a(v) \;\Rightarrow\; \frac{dv}{a(v)} = dt$$

$$\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{t_0}^{t} dt$$

If the acceleration is a function of displacement $s$, use chain rule and separation of variables:

$$a(s)ds = vdv$$

$$\int_{s_0}^{s} a(s)ds = \int_{v_0}^{v} vds$$

## Cartesian coordinates

$$\mathbf{r} = x\mathbf{i} + yj + z\mathbf{k}$$

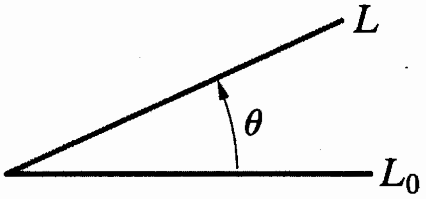
$$\mathbf{v} = v_x\mathbf{i} + v_y j + v_z\mathbf{k}$$

$$\mathbf{a} = a_x\mathbf{i} + a_y j + a_z\mathbf{k}$$

## Angular Motion

$$\omega = \frac{d\theta}{dt}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$



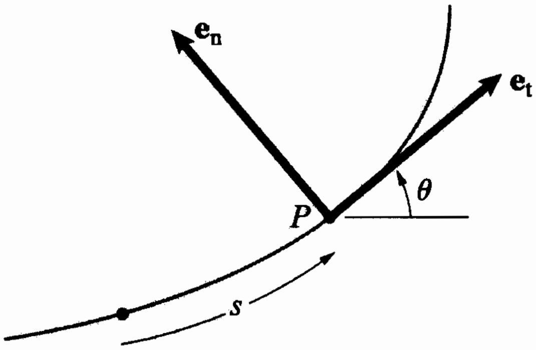
(b)

## Normal and Tangential Components

$$\mathbf{v} = \frac{ds}{dt}\mathbf{e}_t = v\mathbf{e}_t$$

$$\mathbf{a} = a_t\mathbf{e}_t + a_n\mathbf{e}_n$$

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + v\frac{d\theta}{dt}\mathbf{e}_n \qquad \boxed{\mathbf{a} = \frac{dv}{dt}\mathbf{e}_t + \frac{v^2}{\rho}\mathbf{e}_n}$$



(c)

## Polar Coordinates

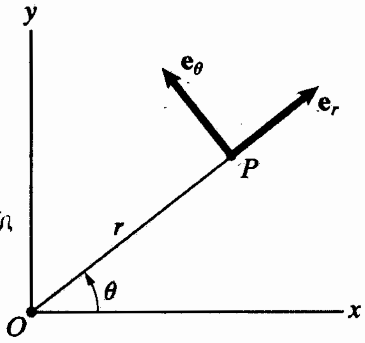
$$\mathbf{r} = r\mathbf{e}_r$$

$$\mathbf{v} = v_r\mathbf{e}_r + v_\theta\mathbf{e}_\theta \qquad \mathbf{v} = \frac{dr}{dt}\mathbf{e}_r + \mathbf{r}\frac{d\mathbf{e}_r}{dt} = \boxed{\frac{dr}{dt}\mathbf{e}_r + \mathbf{r}\omega\mathbf{e}_\theta}$$

$$\mathbf{a} = a_r\mathbf{e}_r + a_\theta\mathbf{e}_\theta$$

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{d^2r}{dt^2} - r\omega^2$$ — centripetal acceleration

$$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = r\alpha + 2\frac{dr}{dt}\omega$$ — Coriolis acceleration



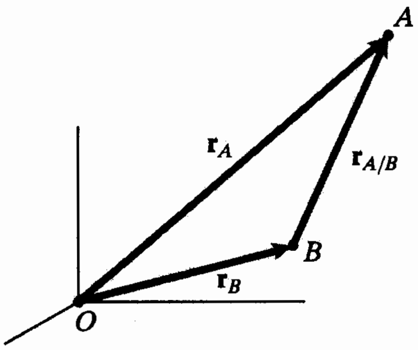
(d)

## Relative Motion

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

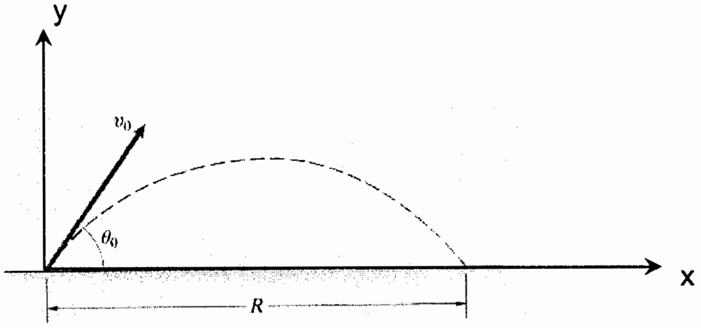
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$



(e)

**2. Trajectory in gravitational field**



**In terms of a fixed Cartesian coordinate system with its y-axis upward:**

$$a_x = 0 \Longrightarrow v_x = v_0 \cos\theta_0 \Longrightarrow x = x_o + v_O \cos\theta_O t$$

$$a_y = -g \Longrightarrow v_y = v_O \sin\theta_O - gt \Longrightarrow y = y_0 + v_o \sin\theta_o t - \frac{1}{2} g t^2$$

**Example 1 (13.81)**

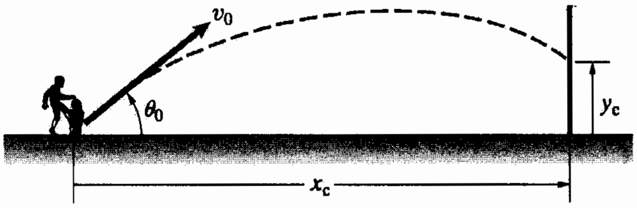
**Knowing:**

$y_c = 10\,ft$

$x_c = 120\,ft$

$v_0 = 70\,ft/s$

$\theta_0 = 70^\circ$



**Find:**
By what vertical distance does the ball clear the crossbar?

Soln

Set the coordinate origin at the point where the ball is kicked.

The x (horizontal) motion of the ball is given by:

$$a_x = 0$$

$$v_x = v_0\cos\theta_0 \Rightarrow x = (v_0\cos\theta_0)t \quad \text{—} ①$$

The y motion is given by:

$$a_y = -g$$

$$v_y = v_0\sin\theta_0 - gt \Rightarrow y = (v_0\sin\theta_0)t - \frac{gt^2}{2} \quad \text{—} ②$$

Set $x = x_c = 120$ ft in ①

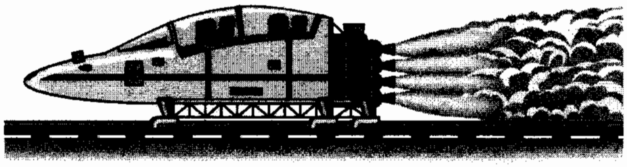
$$t = \frac{120}{70\cos 70} = 2.24\,s$$

Subs. in ②

$$y = (70\sin 70)\,2.24 - 32.2\,\frac{(2.24)^2}{2} = 20.06\ ft$$

Thus the ball clears the cross bar by 10.06 ft.

**Example 2 (13.50)**



**Knowing:**

The rocket sled starts from rest the accelerate with

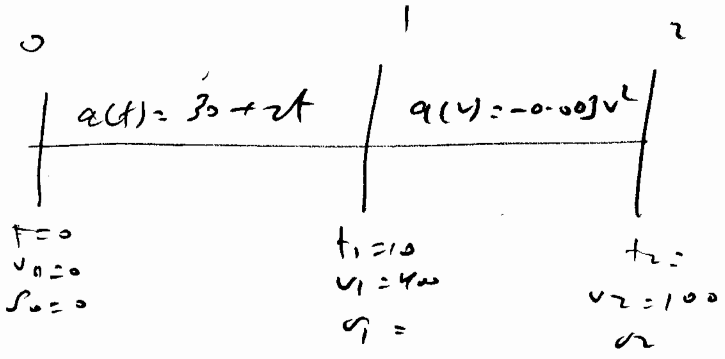
$a = 30 + 2t \quad (m/s^2)$ *until* $v = 400 m/s$

Then it hits a water brake and its acceleration is

$a = -0.003v^2 \quad (m/s^2)$ until its velocity decreases to $100\,(m/s)$.

**Find:**

(a). Total distance the sled travel.



Soln

$a = 30 + 2t$

$v = 30t + t^2$

$s = 15t^2 + \frac{t^3}{3}$

When $v = 400$ m/s, acceleration ends.

At this point $t = 10$ s, $s = 1833$ m.

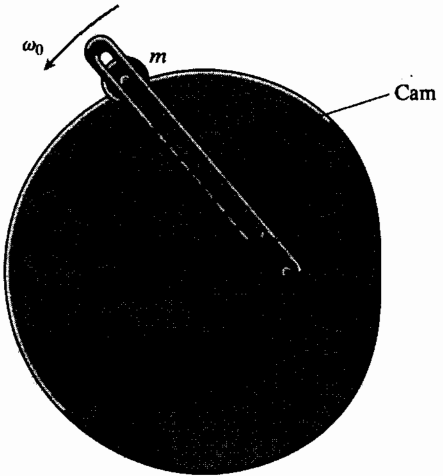
Deceleration phase starts at $s_1 = 1833$ m, $v_1 = 400$ m/s. Let us start a new clock for the deceleration phase. $v_f = 100$ m/s

$$a = v\frac{dv}{ds} = -0.003v^2$$

$$\int_{s_1}^{s_f} ds = -\frac{1}{0.003}\int_{v_1}^{v_f}\frac{v\,dv}{v^2}$$

$$s_f - 1833\text{ m} = -\frac{1}{0.003}\left[\ln(100) - \ln(400)\right] \Rightarrow s_f = 2300\text{ m}$$

**Example 3 (14.107)**



**Knowing:**

Angular velocity (constant) $= \omega_0$

Mass $= m$

$r = r_0(2 - \cos\theta)$

**Find:**

The polar components of the total external force exerted on the pin as a function of $\theta$.

S.ln

The angular velocity is constant, from which

$\theta = \int w_0 dt + C = w_0 t + C$. Assume that $\theta(t=0) = 0 \Rightarrow C = 0$

The radial acceleration is $a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$.

$\frac{d\theta}{dt} = \frac{d}{dt}(w_0 t) = w_0$, $\frac{d^2\theta}{dt^2} = 0$, $\frac{dr}{dt} = \frac{d}{dt}(r_0(2 - \cos\theta))$

$= r_0 \sin\theta\left(\frac{d\theta}{dt}\right) = w_0 r_0 \sin\theta$

$\frac{d^2r}{dt^2} = \frac{d}{dt}(w_0 r_0 \sin\theta_0) = w_0^2 r_0 \cos\theta$

$\therefore a_r = w_0^2 r_0 \cos\theta - r_0(2 - \cos\theta) w_0^2 = 2 r_0 w_0^2 (\cos\theta - 1)$.

From newton second law, The radial component of external force is

$F_r = m a_r = 2 m r_0 w_0^2 (\cos\theta - 1)$

The transverse component of The acceleration is $a_\theta$

$a_\theta = r\frac{d^2\theta}{dt^2} + 2\left(\frac{dr}{dt}\right)\left(\frac{d\theta}{dt}\right)$. substitute;

$a_\theta = 2 r_0 w_0^2 \sin\theta$.

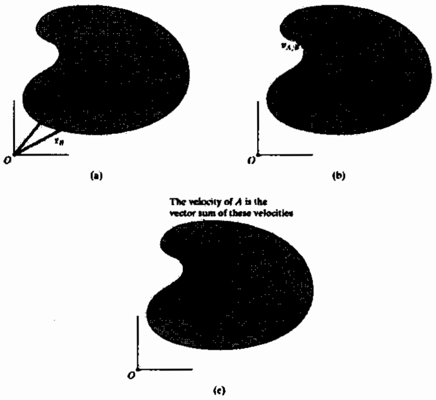
From Newton's second law, The transverse component of external force is $F_\theta = 2 m r_0 w_0^2 \sin\theta$

# Kinematics of Planar Motion of Rigid Bodies

## Relative Velocities

$$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r}_{A/B}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$



## Velocity Formulae

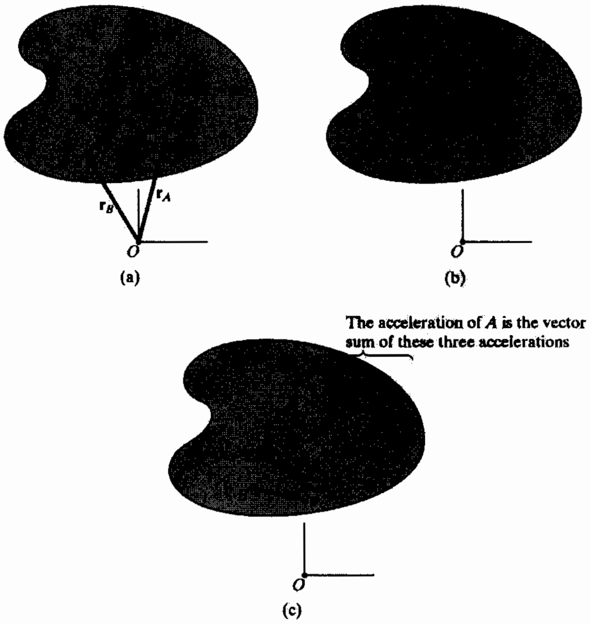
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

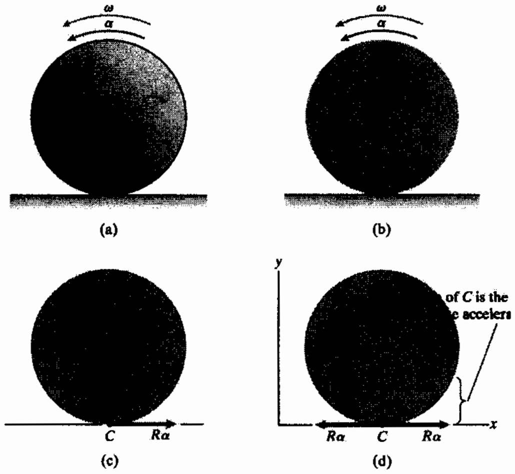
$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}_{A/B}$$

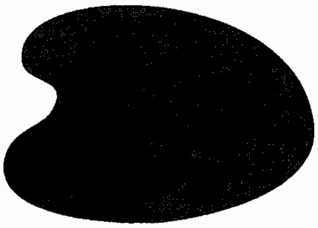
## Acceleration Formulae

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

$$\mathbf{a}_{A/B} = \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{A/B})$$



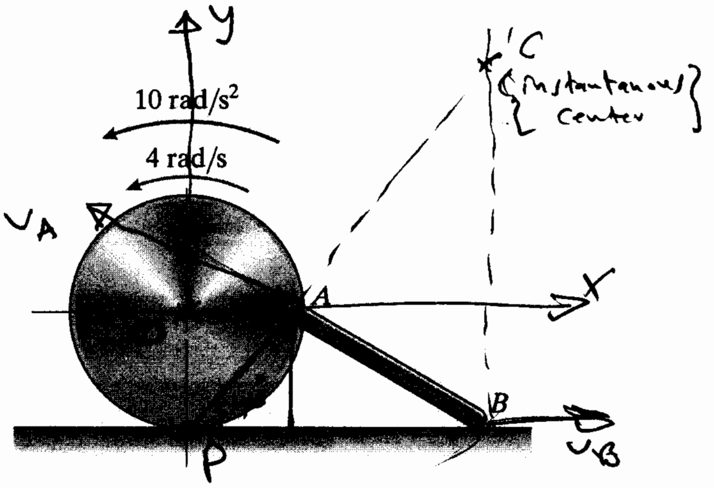




## Planar motion

$$\mathbf{a}_{A/B} = \boldsymbol{\alpha} \times \mathbf{r}_{A/B} - \omega^2 \mathbf{r}_{A/B}$$

**Example 4(17.91)**



## Knowing

The 1-m diameter disk rolls, and point B of the 1-m-long bar slides on the plane surface

## Find

The angular acceleration of the bar and the acceleration of point B.

Sol'n

Choose a coordinate system with the origin at (O).

$$\hat{V}_P = 0 = \hat{V}_O + \hat{\omega}_O \times \underline{R} = \underline{V}_O + \begin{vmatrix} i & j & k \\ 0 & 0 & \omega_O \\ 0 & -0.5 & 0 \end{vmatrix} = \underline{V}_O + 2\hat{i}$$

$$\therefore \underline{V}_O = -2\underline{i} \text{ m/s.}$$

$$\underline{V}_A = \underline{V}_O + \underline{\omega}_O \times \underline{r}_{A/O} = -2\hat{i} + \begin{vmatrix} i & j & k \\ 0 & 0 & \omega_O \\ 0.5 & 0 & 0 \end{vmatrix} = -2\underline{i} + 2\underline{j} \text{ m/s}$$

From the Figure, C is the instantaneous center for link (AB) (at this instant)

$$\underline{r}_{A/C} = -0.866\hat{i} - 0.866\hat{j}$$

$$\underline{V}_A = \hat{\omega}_{AB} \times \underline{r}_{A/C} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega_{AB} \\ -0.866 & -0.866 & 0 \end{vmatrix} = 0.866\omega_{AB}\hat{i} - 0.866\omega_{AB}\hat{j}$$

$$\Rightarrow \omega_{AB} = -\frac{2}{0.866} = -2.31 \text{ rad/s}$$

The acceleration of the center of the rolling disk is $\hat{a}_O = -\alpha R\hat{i}$

$$\hat{a}_O = -10(0.5)\hat{i} = -5\underline{i} \text{ m/s}^2$$

$$\hat{a}_A = \hat{a}_O + \underline{\alpha}_O \times \underline{r}_{A/O} - \omega_O^2 \underline{r}_{A/O} = -5\hat{i} + \begin{vmatrix} i & j & k \\ 0 & 0 & \alpha \\ 0.5 & 0 & 0 \end{vmatrix} - 16(0.5)\hat{i}$$

$$= -13\hat{i} + 5\hat{j} \text{ (m/s}^2\text{)}$$

$$\underline{r}_{B/A} = 0.866i - 0.5j \text{ m}$$

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{\alpha}_{AB} \times \underset{\sim}{r}_{A/B} - \omega^2_{AB}\, \underset{\sim}{r}_{A/B}$$

$$= -13\underset{\sim}{i} + 5\underset{\sim}{j} + \begin{vmatrix} i & j & k \\ 0 & 0 & \alpha_{AB} \\ -\cos\theta & \sin\theta & 0 \end{vmatrix} - \omega^2_{AB}\left(-\underset{\sim}{i}\cos\theta + \underset{\sim}{j}\sin\theta\right)$$

X-direction

$$a_B = -13 + 0.5\alpha_{AB} - \omega^2_{AB}(0.866)$$

Y-direction

$$0 = 5 + 0.866\alpha_{AB} + 0.5\,\omega^2_{AB}$$

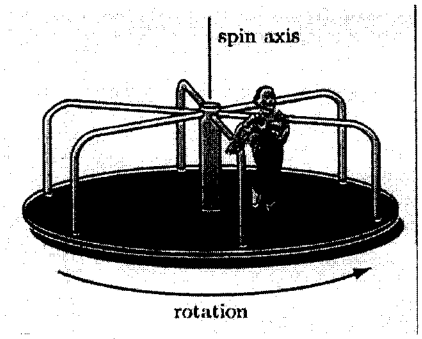
Solve: $\alpha_{AB} = -8.85$ rad/s²

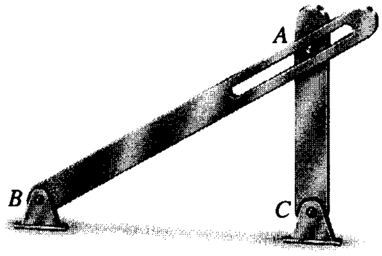
$\underset{\sim}{a}_B = -22.04\underset{\sim}{i}$ m/s²

/9

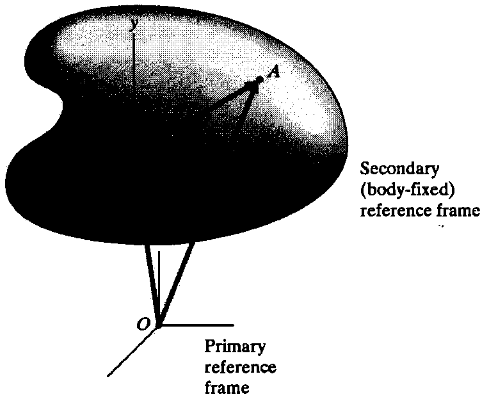
# Sliding Contact

**Examples of "sliding contact"**



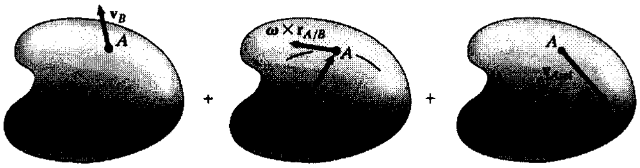


"Sliding contact": motion of an object (point) relative to a *rotating* rigid body.



$$\mathbf{r}_A = \mathbf{r}_B + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A_{rel}} + \mathbf{w} \times \mathbf{r}_{A/B}$$



$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A_{rel}} + 2\mathbf{w} \times \mathbf{v}_{A_{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} + \mathbf{w} \times \left(\mathbf{w} \times \mathbf{r}_{A/B}\right)$$

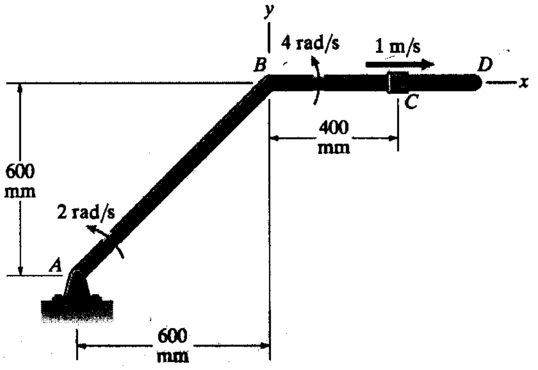
**For planar motion:**

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A_{rel}} + 2\mathbf{w} \times \mathbf{v}_{A_{rel}} + \boldsymbol{\alpha} \times \mathbf{r}_{A/B} - \omega^2 \mathbf{r}_{A/B}$$

**Example 5 (17.120)**

**Knowing**
The angular accelerations of the two bars are zero and sleeve C slides at a constant velocity of 1 m/s relative to bar BD.

**Find**
The acceleration of sleeve C.



Soln

$$\hat{v}_B = \underline{v}_A + \hat{\omega}_{AB} \times r_{\underline{B}/A} = \begin{vmatrix} i & j & k \\ 0 & 0 & 2 \\ 600 & 600 & 0 \end{vmatrix} = -1200(\underline{i}-\underline{j}) \text{ mm/s}$$

$$V_{\underline{C}} = v_{\hat{B}} + v_{Arel} + \omega_{BD} \times r_{\hat{C}/B}$$

$$\underline{V}_C = -1200\hat{i} + 1200\hat{j} + 1000\hat{i} + \begin{vmatrix} i & j & k \\ 0 & 0 & 4 \\ 400 & 0 & 0 \end{vmatrix} = -200\hat{i} + 2800\hat{j} \text{ mm/s}$$

$$\hat{a}_B = \cancel{\underline{a}_A} + \cancel{\underline{a}_{Brel}} + \cancel{\alpha_{AB}} \times r_{\underline{B}|A} - \omega^2_{AB}\, r_{\underline{B}|A} = -\omega^2_{AB}\, r_{\underline{B}/A} = -2400\hat{i} - 2400\hat{j} \text{ mm/s}$$

$$\hat{a}_C = a_B + \underline{a}_{Crel} + 2\hat{\omega}_{BD} \times \underline{V}_{Crel} + \alpha_{BD} \times r_{\underline{C}/B} - \omega^2_{BD}\, \hat{r}_{C/B}$$

$$\hat{a}_C = -2400\hat{i} - 2400\hat{j} + 2\begin{vmatrix} i & j & k \\ 0 & 0 & \omega_{BD} \\ 1000 & 0 & 0 \end{vmatrix} - \omega^2_{BD}(400)\hat{i}$$

$$\hat{a}_C = -8800\hat{i} + 5600\hat{j} \text{ mm/s}^2$$

# Newton's Second Law

The total force on a particle is equal to the rate of change of its linear momentum, which is the product of its mass and velocity.

**Key concepts:**

$\sum \mathbf{F} = m\mathbf{a}$

- **Cartesian coordinates**
- $\sum F_x = ma_x \quad \sum F_y = ma_y \quad \sum F_z = ma_z$

- **Normal and tangential components**
- $\sum F_t = m\frac{dv}{dt} \quad \sum F_n = m\frac{v^2}{\rho}$

- **Polar coordinates**

$$\sum F_r = m\left(\frac{d^2r}{dt^2} - r\omega^2\right) \quad \sum F_\theta = m\left(r\alpha + 2\frac{dr}{dt}\omega\right)$$

# Kinetics of Planar Motion of Rigid Bodies

## Moment-Angular-Momentum Relations

$$\sum \mathbf{M}_O = \frac{d\mathbf{H}_O}{dt}$$

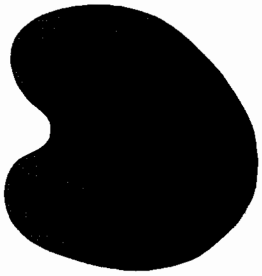
$$\mathbf{H}_O = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i$$

$$\sum \mathbf{M}_O = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v} + \mathbf{H})$$

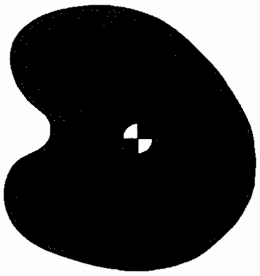
$$\mathbf{H} = \sum_i \mathbf{R}_i \times m_i \frac{d\mathbf{R}_i}{dt}$$

$$\sum \mathbf{M} = \frac{d\mathbf{H}}{dt}$$

$$\mathbf{H}_O = (\mathbf{r} \times m\mathbf{v} + \mathbf{H})$$



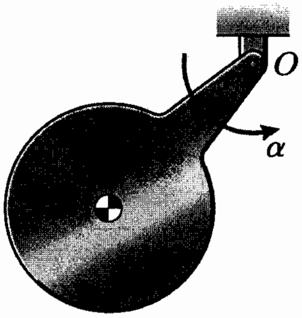
(a)



(b)

## Equations of planar motion

- Translation
  $$\sum \mathbf{F} = m\mathbf{a}$$
  $\mathbf{a}$ : acceleration of the *mass center*
- Rotation (about a fixed axis)
  $$\sum \mathbf{M}_o = I_o \alpha$$
  $\alpha$ : angular acceleration of the body;
  $I_o$ : moment of inertia about point O.
- Rotation (around the axis passing through the mass center
  $$\sum \mathbf{M} = I\alpha$$
  $\alpha$ : angular acceleration of the body;
  $I$ : moment of inertia about the mass center.



$$\Sigma M_O = I_O \alpha$$

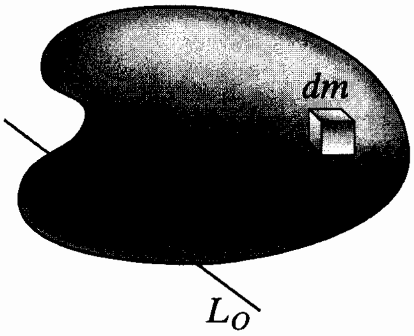
## Moments of Inertia

The moment of inertia of an object about an axis $L_O$ is

$$I_O = \int_m r^2 dm$$

For an axis $L$ parallel to $L_O$, we can use the parallel-axis theorem

$$I_O = I + d^2 m$$

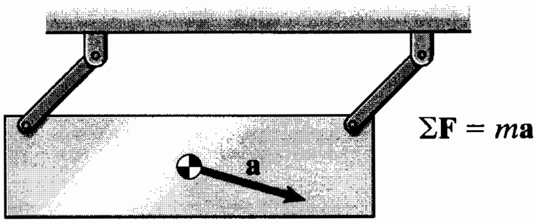


(c)

where m is the mass of the object and d is the distance between $L$ and $L_O$.

**Translation**

$\sum M = 0$



Note

In general planar Motion, The number of unknows is usually more than ③. So, we need to use equation of motion (Kinematic) in addition to The Kinetics equations.

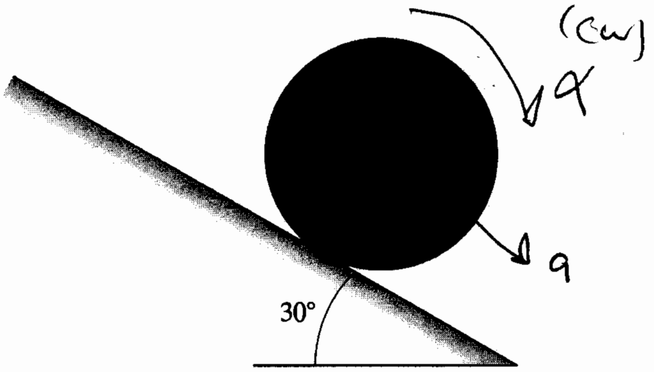
**Example 6 (18.33)**

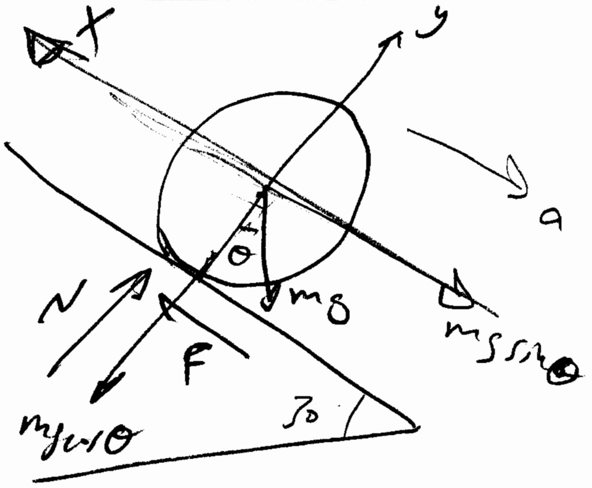
**Knowing**

The radius of the 2kg disk is R=80mm. It is released from rest on the rough inclined surface.

**Find**

The time the disk takes to roll through one revolution.





$\Sigma F_x = m a_x$

$F - mg\sin\theta = -m a_y$

$\Sigma F_y = m a_y$

$N - mg\cos\theta = 0$

$\Sigma M_G = I\alpha$

$-F \cdot R = -\frac{1}{2} m R^2 \alpha$

$a_x = R\alpha$

$F = \mu N$

solving we find:

$\alpha = \frac{g}{3R}$ ; $a = \frac{g}{3}$ ; $N = 0.867\,mg$, $F = 0.167\,mg$, $\mu = 0.192$

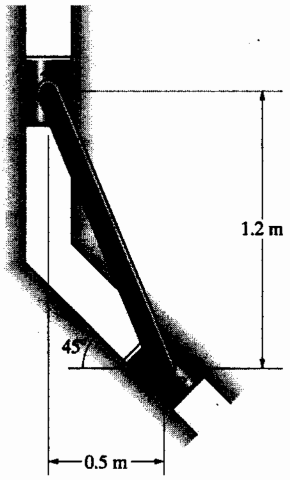
$\alpha = 40.9\ \text{rad/s}^2 \Rightarrow \omega = \alpha t \Rightarrow \theta = \frac{1}{2}\alpha t^2$

For one revolution $(\theta = 2\pi) \Rightarrow t = 0.554\ s.$

**Example 7 (18.67)**

**Knowing**
The 4kg slender bar is pinned to 2kg sliders at A and B. If friction is negligible and the system is released from rest in the position shown,

**Find**
The angular acceleration of the bar at that instant.



Soln

$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \underset{\sim}{\alpha}_{AB} \times \underset{\sim}{r}_{B/A} - \cancel{\omega^2_{AB}}\, \underset{\sim}{r}_{B/A}$

$= \underset{\sim}{a}_A + \underset{\sim}{\alpha}_{AB} \times \underset{\sim}{r}_{B/A}$.

$a_B \cos 45 \underset{\sim}{i} - a_B \sin 45° \hat{j} = -a_A \hat{j} + \begin{vmatrix} i & j & k \\ 0 & 0 & \alpha_{AB} \\ 0.5 & -1.2 & 0 \end{vmatrix}$

or $a_B \cos 45° = 1.2\alpha_{AB}$ —— ①

and $a_B \sin 45 = -a_A + 0.5\alpha_{AB}$ —— ②

$\underset{\sim}{a}_G = \underset{\sim}{a}_A + \underset{\sim}{\alpha}_{AB} \times \underset{\sim}{r}_{G/A}$

$\underset{\sim}{a}_G = a_{Gx} \hat{i} + a_{Gy} \underset{\sim}{j} = -a_A \hat{j} + \begin{vmatrix} i & j & k \\ 0 & 0 & \alpha_{AB} \\ 0.25 & -0.6 & 0 \end{vmatrix}$

or $a_{Gx} = 0.6\alpha_{AB}$ —— ③

and $a_{Gy} = -a_A + 0.25\alpha_{AB}$ —— ④

The free body diagrams are as shown,

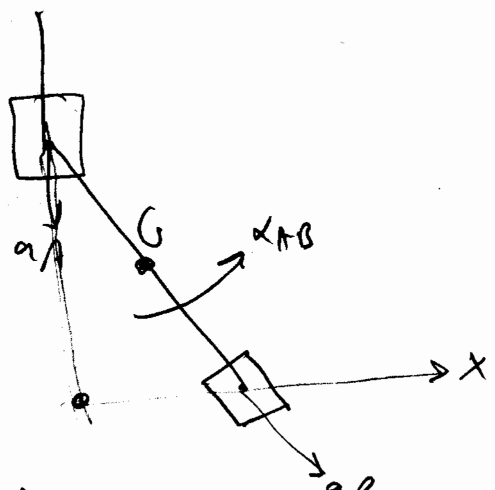
For slider A.

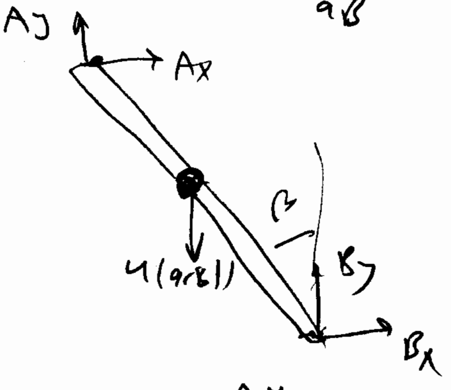
$\Sigma F_x = m a_x = 0$

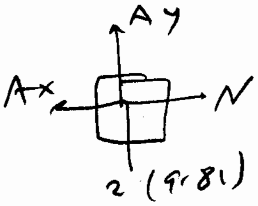
$N - A_x = 0$ —— ⑤

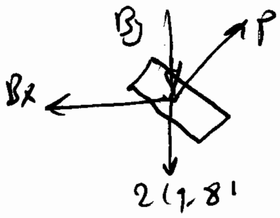
$\Sigma F_y = m a_{Ay}$

$2(9.81) + A_y = 2 a_{Ax}$ —— ⑥









## Slider B

$P - [B_x + B_y + (2)(9.81)]\cos 45^\circ = 0$ —— ⑦ ; $\Sigma F_x = ma_x$

and

$[(2)(9.81) - B_x + B_y]\cos 45^\circ = 2a_B$ —— ⑧ ; $\Sigma F_y = ma_y$

## Bar AB

$A_x + B_x = 4a_{Gx}$ —— ⑨ ; $\Sigma F_x = ma_x$

$A_y + B_y - 4(9.81) = 4a_{Gy}$ —— ⑩ ; $\Sigma F_y = ma_y$

$(L/2)[(B_x - A_x)\cos\beta + (B_y - A_y)\sin\beta] = \frac{1}{12}(4)L^2\alpha_{AB}$ —— ⑪ ; $\Sigma M = I\alpha$

where

$L = \sqrt{(0.5)^2 + (1.2)^2}$ m

$\beta = \tan^{-1}\left(\frac{0.5}{1.2}\right) = 22.6^\circ$

solving eqns (1)-(11) $\longrightarrow \alpha_{AB} = 5.18$ rad/s²

[redacted]

**Basic Concepts** ( Particles )

- **Principle of Work and Energy**

$$U_{12} = \int_{r_1}^{r_2} \sum \mathbf{F}.d\mathbf{r}$$

$$U_{12} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

- **Power**

$$P = \sum \mathbf{F}.\mathbf{v}$$

$$P_{av} = \frac{\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2}{t_2 - t_1} = \frac{U_{12}}{t_2 - t_1}$$

- **Evaluating the Work**

$$U_{12} = \int_{s1}^{s2} \sum F_t ds$$

$$U_{12} = -mg(y_2 - y_1) \qquad \text{(Weight)}$$

$$U_{12} = mgR_E^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$U_{12} = -\frac{1}{2}k(S_2^2 - S_1^2) \qquad \text{(Springs)}$$

- **Potential Energy**

$$dV = -\sum \mathbf{F}.d\mathbf{r}$$

**If all the forces are conservative then:**

$$\frac{1}{2}mv_1^2 + V_1 = \frac{1}{2}mv_2^2 + V_2$$

**If the object is subjected to both conservative and nonconservative forces, the principle of work and energy can be written:**

$$\frac{1}{2}mv_1^2 + V_1 + U_{12} = \frac{1}{2}mv_2^2 + V_2$$

$$V = mgy \qquad \text{(Weight)}$$

$$V = -\frac{mgR_E^2}{r}$$

$$V = \frac{1}{2}kS^2 \qquad \text{(Springs)}$$

**Relationships between Force and Potential Energy**

$$\mathbf{F} = -\left(\frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k}\right) = -\nabla V$$

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

The force $\underline{F}$ is conservative if $\underline{\nabla} \times \underline{F} = \underline{0}$

# Work and Energy

**Work and Energy** (Rigid bodies)

$$U_{12} = T_2 - T_1$$

The kinetic energy of a rigid body in general planar motion is:

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

where $v$ is the magnitude of the velocity of the center of mass of the body

If a rigid body rotates about a fixed axis O, its kinetic energy can also be expressed as

$$T = \frac{1}{2}I_O\omega^2$$

The work done on a rigid body by a force **F** is:

$$U_{12} = \int_{(r_p)_1}^{(r_p)_2} \mathbf{F}.d\mathbf{r}_p$$

where $\mathbf{r}_p$ is the position of the point of application of **F.**

The work done by a couple *M* on a rigid body in planar motion as the body rotates from $\theta_1$ to $\theta_2$ in the direction of *M* is:

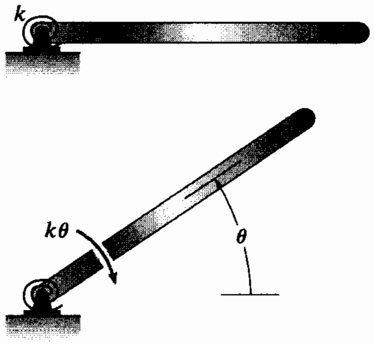
$$U_{12} = \int_{\theta_1}^{\theta_2} M d\theta$$

The couple *M* is conservative if a potential energy *V* exists such that:

$$Md\theta = -dV$$

If all forces and couples that do work on a system are conservative, then, the sum of kinetic energy and the total potential energy is constant.

$$T_1 + V_1 = T_2 + V_2$$



(a)

If a system is subjected to both conservative and nonconservative forces, then the principle of work and energy can be written in the form

$$T_1 + V_1 + U_{12} = T_2 + V_2$$

**Power**

The power transmitted to a rigid body by a force **F** is:

$$P = \mathbf{F}.\mathbf{v}_P$$

where $\mathbf{v}_p$ is the point of application of **F**. The power transmitted to a rigid body in planar motion by a couple *M* is:

$$P = M\omega$$

$$P_{av} = \frac{T_2 - T_1}{t_2 - t_1} = \frac{U_{12}}{t_2 - t_1}$$

**Impulse and Momentum**

$$\int_{t_1}^{t_2} \sum \mathbf{F} dt = m\mathbf{v}_2 - m\mathbf{v}_1$$

$$(t_2 - t_1)\sum \mathbf{F}_{av} = m\mathbf{v}_2 - m\mathbf{v}_1$$

$$m_A\mathbf{v}_A + m_A\mathbf{v}_A = const$$

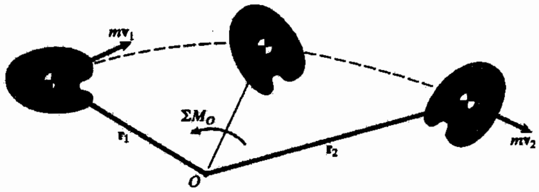
$$H = I\omega$$

where *I* is the moment of inertia about its center of mass.

$$\int_{t_1}^{t_2} \sum M dt = H_2 - H_1$$

$$H_O = (\mathbf{r} \times m\mathbf{v}).\mathbf{k} + I\omega$$

where **r** is the position of the center of mass relative to O and **v** is the velocity of the center of mass



$$(\mathbf{r}_1 \times m\mathbf{v}_1) \cdot \mathbf{k} + I\omega_1 + \int_{t_1}^{t_2} \Sigma M_O \, dt = (\mathbf{r}_2 \times m\mathbf{v}_2) \cdot \mathbf{k} + I\omega_2$$

$$\int_{t_1}^{t_2} \sum M_O dt = H_{O2} - H_{O1}$$

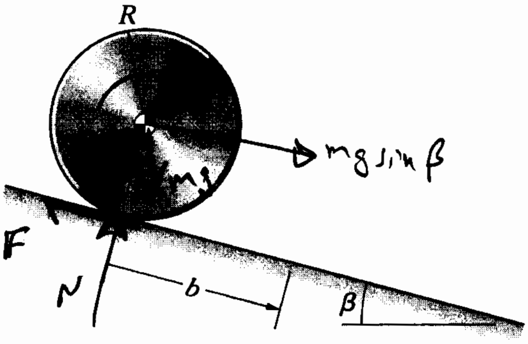
$$(t_2 - t_1)\sum M_{av} = H_2 - H_1$$

$$(t_2 - t_1)\sum M_{O_{av}} = H_{O2} - H_{O1}$$

If the total moment due to external forces and couples about a fixed point O is zero, then

$$H_{OA} + H_{OB} = const$$

**Example 8 (19.20)**



**Knowing**
The mass of the homogenous cylindrical disk is m=5kg and its radius is R=0.2m. The angle $\beta = 15°$. The disk is stationary when a constant clockwise couple M=10N.m is applied to it.

**Find**
The velocity of the center of the disk when it has moved a distance b=0.4m.

Soln

$\theta = \frac{b}{R}$

The work done by the couple & the disk's weight is:

$U_{12} = M\left(\frac{b}{R}\right) + mg\,b\sin\beta$

Applying work & kinetic energy principle. $U_{12} = T_2 - \cancel{T_1}$

$M\left(\frac{b}{R}\right) + mg\,b\sin\beta = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

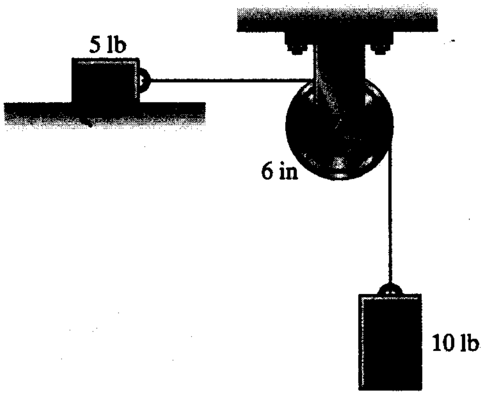
$\omega = v/R$

$v = 2\sqrt{\frac{b}{3}\left(\frac{M}{mR} + g\sin\beta\right)} = 2\sqrt{\frac{0.4}{3}\left[\frac{10}{5(0.2)} + 9.81\sin 15\right.} = 2.59$ m/

**Example 9 (19.25)**

**Knowing**

The two weights are released from rest. The 5lb weight slides on the horizontal surface. The moment of inertia of the pulley is I=0.02 slug.ft$^2$.



**Find**

The magnitude of the velocity of the 10lb weight when it has fallen 2 ft,

(a) No fiction.

(b) Friction with $\mu_K = 0.4$

Soln

(a)

$$T_1 = 0,\ V_1 = 0,\ T_2 = \frac{1}{2}\left(\overset{(10+5)}{\frac{15}{32.2}}\right)v^2 + \frac{1}{2}(0.02)\left(\underbrace{\frac{v}{\frac{6}{12}}}_{\omega}\right)^2$$

$$V_2 = -(10)(2)$$

Applying conservation of energy.

$$T_1 + V_1 = T_2 + V_2 \Rightarrow v = 8.56 \text{ ft/s}$$

(b)

$$T_1 = 0,\ V_1 = 0,\ T_2 = \frac{1}{2}\left(\frac{15}{32.2}\right)v^2 + \frac{1}{2}(0.02)\left(\frac{v}{\frac{6}{12}}\right)^2$$

$$V_2 = (-10)(2),$$

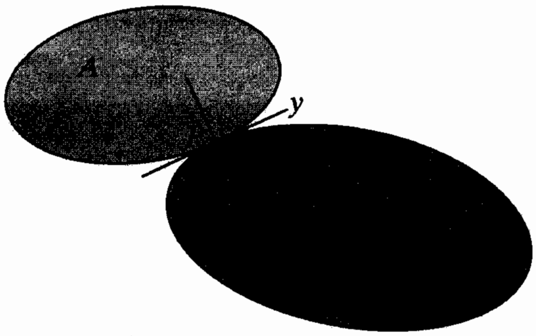
$$U_{12} = -0.4(5)(2)$$

$$T_1 + V_1 + U_{12} = T_2 + V_2 \Rightarrow v = 7.66 \text{ ft/s}$$

## Impacts

Let *P* be the point of impact (figure b). The normal components of the velocities at *P* are related to the coefficient of restitution *e* by

$$e = \frac{(v'_{BP})_x - (v'_{AP})_x}{(v_{AP})_x - (v_{BP})_x}$$

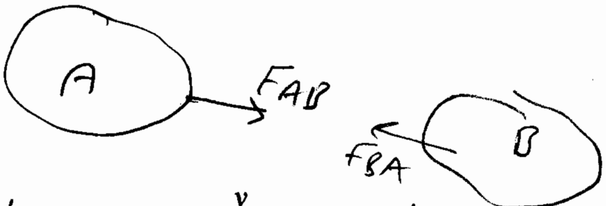


(b)

## Principle of Impulse and Momentum

$$\int_{t_1}^{t_2} \sum \mathbf{F} dt = m\mathbf{v}_2 - m\mathbf{v}_2$$

$$(t_2 - t_1) \sum \mathbf{F}_{av} = m\mathbf{v}_2 - m\mathbf{v}_1$$

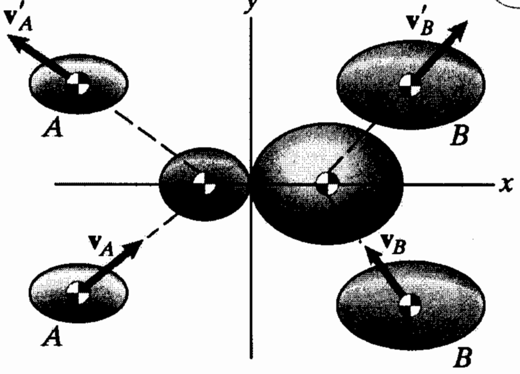


## Conservation of Linear Momentum

$$m_A \mathbf{v}_A + m_B \mathbf{v}_B = const$$

## Impacts

$$\mathbf{v} = \frac{m_A \mathbf{v}_A + m_B \mathbf{v}_B}{m_A + m_B} = const \quad (e = 0)$$



## Central Impacts

**Direct impact:**

$$m_A \mathbf{v}_A + m_B \mathbf{v}_B = m_A \mathbf{v}'_A + m_B \mathbf{v}'_B$$

**Oblique impact:** $\underline{m_A(\mathbf{v}_A)_x} + \underline{m_B(\mathbf{v}_B)_x} = \underline{m_A(\mathbf{v}'_A)_x} + \underline{m_B(\mathbf{v}'_B)_x}$

$$e = \frac{(\mathbf{v}'_B)_x - (\mathbf{v}'_A)_x}{(\mathbf{v}_A)_x - (\mathbf{v}_B)_x}$$

Velocities in tangential plane are not changing.

## Principle of Angular Impulse and Momentum

$$\int_{t_1}^{t_2} (\mathbf{r} \times \sum \mathbf{F})\, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1$$

$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$$

## Central-Force Motion

$$\mathbf{H}_O = const$$

$$\mathbf{H}_O = (r\mathbf{e}_r) \times m(v_r\mathbf{e}_r + v_\theta \mathbf{e}_\theta) = mrv_\theta \mathbf{e}_z$$

## Mass Flows

$$\mathbf{F}_f = -\frac{dm_f}{dt}\mathbf{v}_f$$

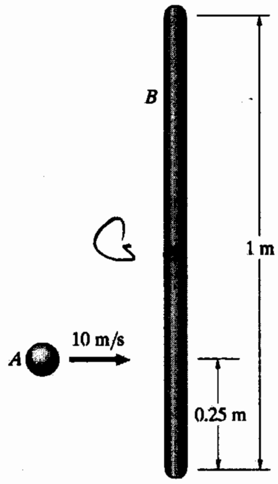
**Example 10 (19.72)**

**Knowing**

The 2kg sphere A is moving at 10m/s when it strikes the stationary unconstrained 4kg bar. The coefficient of restitution is e=0.1.

Find

(a) The bar angular velocity after the impact.
(b) The total kinetic energy of the sphere and bar before and after the impact.

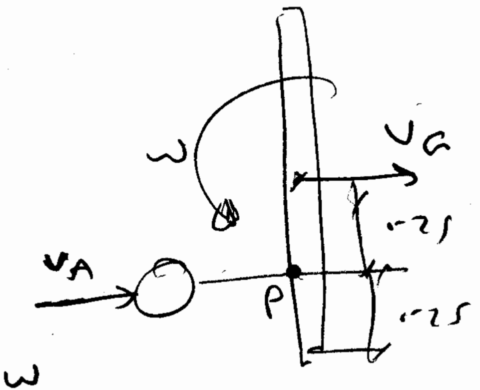


Soln

(a) linear momentum

$m_A v_A + \cancel{m_B v_B}^{0} = m_A \bar{v}_A + m_B \bar{v}_G$

$(2)(10) = 2(v_A) + 4\, v_G$



Angular Momentum about G

$\underline{(r \times mv)\cdot k + I\cancel{\omega}} = (r \times mv)\cdot k + I\omega$

$\underline{(2)(10)(.25)} = (2)\, v_A\, (0.25) + \frac{1}{12}(4)(1)^2 \omega$

Coefficient of restitution $\Rightarrow$ at point of impact P. $e = \dfrac{\bar{v}_{BP} - \bar{v}_{AP}}{v_A - \cancel{v_B}}$

$1\,(10) = v_G + \omega(0.25) - \underline{\underline{v_A}}$ ; where $v_{BP} = v_G + \omega \times r_{B/P}$

Solving, we find $v_A = -0.667$ m/s ; $v_G = 5.33$ m/s

$\omega = 16$ rad/s ccw

(b) $T_{before} = \frac{1}{2}(2)(10)^2 = 100\,N$

$T_{after} = \frac{1}{2}(2)(-0.667)^2 + \frac{1}{2}(4)(5.33)^2 + \frac{1}{2}\left[\frac{1}{12}(4)(1)^2\right] 16 = 100\,N$

$T_{before} = T_{after} = 100\,N{\cdot}m$